Exploring Generating Functions as Random Walks

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Abstract

Generating functions provide convenient tools to understand the behavior of infinite sequences. A more specific form of these functions is a probability generating function, which can also be utilized to find the expected value of a discrete random variable. In our study, we used concepts from probability theory to explore Poisson probability generating functions and used them along with Poisson distributions to model a random walk in a plane. With the use of Euler's Formula, we were able to find where these functions converged, diverged, and the directions they moved with their real and imaginary components. This then leads us to more complex forms of functions and how they can then be translated into more complicated random walks. These random walks can be applied to real life with the study of population genetics, vision science, and the psychology of decision making.

What is a Generating Function?

A generating function is an infinite series that has the ability to be summed, however some may not have this quality. These series act as a filing cabinet where the information stored within can be seen on the coefficient on z^n.

Probability Generating Function

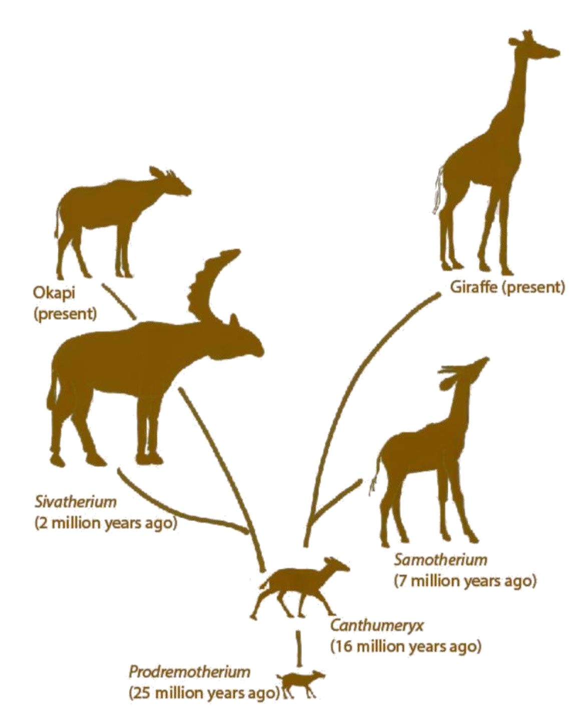
A probability mass function is a list of probabilities for each possible value the random variable can take on. These values can be extracted from a special type of generating function known as a probability generating function (PGF). The PGF is where the values of the coefficients on z^n are the probability the random variable takes the wanted value n.

Poisson Distribution

Poisson distributions show the probability of a number of events happening in a specific time interval. We used them in our project because they can model each step in a random walk, as each step is an independent event.

Random Walks in the Plane

A random walk describes the irregular movements of a particle on a mathematical plane. The plane can consist of integer lattice or something more advanced like the way a brain analyzes the decisions it is about to make. The easiest way to imagine a random walk and the manner they work is with a drunken sailor. While drunk, the sailor has very poor balance so they decide to take a step to the right, but then goes off balance and moves back to the left two. As the night progresses, the sailor, if his steps were tracked, would seem completely random and useless. However with random walks, decision making, vision science, population genetics, and more can all be analyzed.



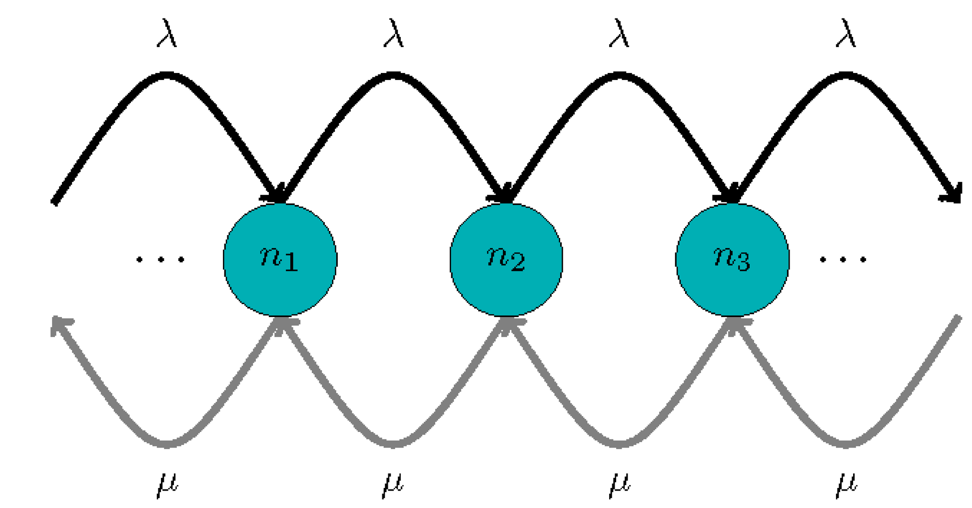
Generating Function for a Random Walk

$$G_N(z) = \left(\sum_{n=0}^{\infty} z^n \frac{\lambda^n}{n!} e^{-\lambda}\right)\left(\sum_{n=0}^{\infty} \frac{\mu^n z^{-n}}{n!} e^{-\mu}\right)$$

$$= e^{\lambda(z-1)+\mu(z^{-1}-1)}$$

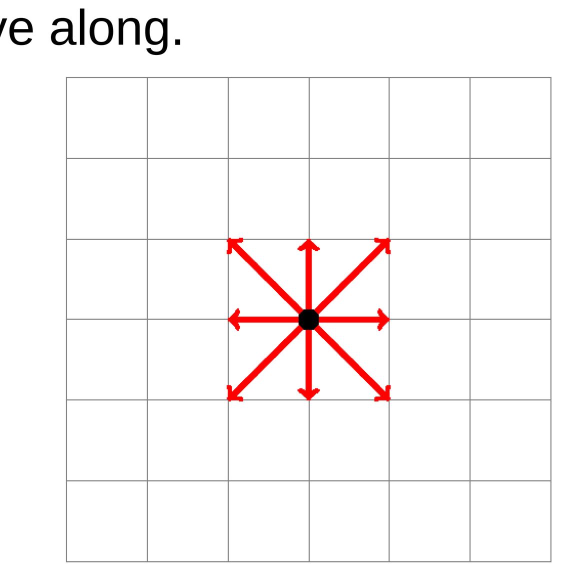
Stochastic Flow Diagram

We used a stochastic flow diagram as visual representation that can be used to show how a random walk can move in certain directions.



$$\dot{p}_n(t) = \lambda p_{n-1} - (\lambda + \mu)p_n(t) + \mu p_{n+1}(t)$$

The diagram above depicts a bidirectional movement; however, we looked at up to eight directions that a random walk could move along.



Conclusion

We explored generating functions and their ability to store information pertaining to the probability of random events, specifically in the case of random walks in the plane. We may continue to research more complex functions and their uses in the future.

References and Acknowledgments

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